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Note

Extreme eigenvalues of nonregular graphs

Sebastian M. Cioabă [a,1], David A. Gregory [b,2], Vladimir Nikiforov [c]

[a] *Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA*
[b] *Department of Mathematics, Queen's University at Kingston, Ontario, K7L 3N6, Canada*
[c] *Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA*

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Abstract

Let λ_1 be the greatest eigenvalue and λ_n the least eigenvalue of the adjacency matrix of a connected graph G with n vertices, m edges and diameter D. We prove that if G is nonregular, then

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \geqslant \frac{1}{n(D+1)},$$

where Δ is the maximum degree of G.

The inequality improves previous bounds of Stevanović and of Zhang. It also implies that a lower bound on λ_n obtained by Alon and Sudakov for (possibly regular) connected nonbipartite graphs also holds for connected nonregular graphs.

Keywords: Spectral radius; Nonregular graph; Eigenvalues

1. Background

Let G be a simple graph with vertex set $[n] = \{1, 2, \dots, n\}$, maximum degree Δ and minimum degree δ. Let λ_1 denote the largest eigenvalue of the adjacency matrix A of G. The Perron–Frobenius Theorem [2, p. 178] implies that λ_1 has an eigenvector x with nonnegative entries

E-mail addresses: scioaba@math.ucsd.edu (S.M. Cioabă), gregoryd@mast.queensu.ca (D.A. Gregory), vnikifrv@memphis.edu (V. Nikiforov).

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which must be positive if G is connected. Then $Ax = \lambda_1 x$ and, so $\lambda_1 \sum_i x_i = \sum_j d_j x_j$ where $d_1, d_2, \dots, d_n$ is the degree sequence of G. This implies the well-known result that

$$\delta \leqslant \lambda_1 \leqslant \Delta$$

and that, when G is connected, equality holds in either case if and only if G is regular.

Stevanović [3] proved that if a connected graph G is nonregular, then

$$\Delta - \lambda_1 > \frac{1}{2n(n\Delta - 1)\Delta^2}.$$

In the final remark of [3], Stevanović asked whether or not the power of Δ could be improved. Recently, Zhang [4] obtained the finer bound

$$\Delta - \lambda_1 > \frac{(\sqrt{\Delta} - \sqrt{\delta})^2}{nD\Delta},$$

where D is the diameter of G. In this note, we prove the following stronger inequality for a connected nonregular graph G:

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}. \tag{1}$$

The Perron–Frobenius Theorem also implies that if λ_n is the least eigenvalue of a graph, then $\lambda_1 \geqslant -\lambda_n$ with equality in the connected case if and only if the graph is bipartite [2, p. 178]. For graphs that are connected and nonbipartite (and possibly regular), Alon and Sudakov [1] proved that

$$\Delta + \lambda_n > \frac{1}{n(D+1)}. \tag{2}$$

Because the lower bound in (1) is monotone increasing in $n\Delta - 2m$, it follows that, for a connected nonregular graph G,

$$\Delta + \lambda_n \geqslant \Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \geqslant \frac{1}{n(D+1)}. \tag{3}$$

Thus, the bound (2) of Alon and Sudakov also holds for connected nonregular graphs and, in that case, is refined by the inequalities (3).

2. The proof

In this section, we present a proof of inequality (1).

Theorem 1. *If G is a connected nonregular graph with diameter D, then*

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}.$$

Proof. Let x be the unique unit positive eigenvector of A with eigenvalue λ_1. Let E denote the edge set of G and, for each $i \in [n]$, let d_i denote the degree of vertex i. Then

$$\lambda_1 = x^t Ax = 2\sum_{ij \in E} x_i x_j = \sum_{ij \in E} \left(x_i^2 + x_j^2\right) - \sum_{ij \in E} (x_i - x_j)^2 = \sum_{i=1}^{n} d_i x_i^2 - \sum_{ij \in E} (x_i - x_j)^2.$$

This implies

$$\Delta - \lambda_1 = \sum_{ij \in E} (x_i - x_j)^2 + \sum_{i=1}^{n} (\Delta - d_i) x_i^2. \tag{4}$$

Choose vertices $s, t \in [n]$ so that $x_s = \max_i x_i$ and $x_t = \min x_i$. Then $t \neq s$ because G is nonregular. If $s = i_0, i_1, \ldots, i_{k-1}, i_k = t$ are consecutive vertices of a shortest path from s to t in G, it follows from the Cauchy–Schwarz inequality that

$$\sum_{j=0}^{k-1} (x_{i_j} - x_{i_{j+1}})^2 \geqslant \frac{1}{k} \left(\sum_{j=0}^{k} (x_{i_j} - x_{i_{j+1}}) \right)^2 = \frac{1}{k} (x_s - x_t)^2 \geqslant \frac{1}{D} (x_s - x_t)^2.$$

Thus, from (4) we obtain

$$\Delta - \lambda_1 \geqslant \frac{(x_s - x_t)^2}{D} + (n\Delta - 2m) x_t^2,$$

where the right-hand side is a quadratic function of x_t that attains its minimum when $x_t = \frac{x_s}{D(n\Delta - 2m) + 1}$. It follows that

$$\Delta - \lambda_1 \geqslant \frac{(n\Delta - 2m) x_s^2}{D(n\Delta - 2m) + 1},$$

where $x_s^2 > \frac{1}{n}$ since $x^T x = 1$. The statement in Theorem 1 now follows. $\square$

A further refinement of Theorem 1 may be possible. For example, a computer search shows that $\Delta - \lambda_1 > (\Delta - \delta)^{\frac{1}{2}} / nD$ for all connected nonregular graphs of order at most 8.

3. Examples

The lower bound on $\Delta - \lambda_1$ in Theorem 1 is of order $O(\frac{1}{nD})$. The bipartite graphs in the following proposition suggest that for connected nonregular graphs with $\Delta - \delta = 1$, this order is best possible.

Let $G_1, \ldots, G_k$ be k disjoint copies of the complete bipartite graph $K_{\Delta,\Delta}$. Remove an edge, say $v_{2i-1} v_{2i}$, from each G_i, and join v_{2i} to v_{2i+1} for each $i = 1, \ldots, k-1$. Let $G_{\Delta,k}$ denote the resulting chain of bipartite graphs. Clearly $G_{\Delta,k}$ has $n = 2k\Delta$ vertices, maximum degree Δ, minimum degree $\delta = \Delta - 1$ and diameter $D = 4k - 1$.

Proposition 2. *For the graphs $G_{\Delta,k}$, $\Delta - \lambda_1 < \frac{4\pi^2}{nD}$.*

Proof. Let A be the adjacency matrix of $G_{\Delta,k}$ and let $U = \{v_1, v_{2k}\}$, the two vertices of degree $\delta = \Delta - 1$. For each unit vector z, because $\lambda_1 \geqslant z^T A z$, the argument used to prove (4) gives

$$\Delta - \lambda_1 \leqslant \sum_{ij \in E} (z_i - z_j)^2 + \sum_{i \in V} (\Delta - d_i) z_v^2 = \sum_{ij \in E} (z_i - z_j)^2 + \sum_{i \in U} z_i^2. \tag{5}$$

Let y be the unit eigenvector corresponding to the greatest eigenvalue $2\cos\frac{\pi}{k+1}$ of the path P_k on k vertices. By (4),

$$\sum_{i=1}^{k-1} (y_i - y_{i+1})^2 + y_1^2 + y_k^2 = 2 - \lambda_1(P_k) < \frac{\pi^2}{(k+1)^2}. \tag{6}$$

For $v \in V(G_{\Delta,k})$, let $z_v = y_i/\sqrt{2\Delta}$ when $v \in G_i$. Then $z^T z = 1$. Substituting z into (5), we obtain

$$\Delta - \lambda_1 \leqslant \sum_{i=1}^{k-1}\left(\frac{y_i - y_{i+1}}{\sqrt{2\Delta}}\right)^2 + \frac{y_1^2}{2\Delta} + \frac{y_k^2}{2\Delta} < \frac{\pi^2}{2\Delta(k+1)^2} < \frac{4\pi^2}{nD}. \qquad \square$$

We noted above that Theorem 1 implies that there is a constant c such that $\Delta - \lambda_1 > \frac{c}{nD}$ for all connected nonregular graphs. It is easy to check that the theorem implies that $c \geqslant 2/3$. Taking a graph formed from the binary tree on 7 vertices, together with a 4-cycle on the vertices of degree 1, we get $c \leqslant nD(\Delta - \lambda_1) \approx 1.355$. We conjecture that $c \geqslant 1$ or, equivalently, that $\Delta - \lambda_1 > 1/nD$ for all connected nonregular graphs.

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